Non-Abelian BFFT embedding, Schrödinger quantization and the anomaly of the $O(N)$ nonlinear sigma model

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Abstract

We embed the $O(N)$ nonlinear sigma model in a non-Abelian gauge theory. As a first class system, it is quantized using two different approaches: the functional Schrödinger method and the non-local field-antifield procedure. Firstly, the quantization is performed with the functional Schrödinger method, for $N = 2$, obtaining the wave functionals for the ground and excited states. In the second place, using the BV formalism we compute the one-loop anomaly. This important result shows that the classical gauge symmetries, appearing due to the conversion via BFFT method, are broken at the quantum level.

I. INTRODUCTION

Batalin, Fradkin, Fradkina, and Tyutin (BFFT) [1,2] developed an elegant formalism for embedding second class systems in first class ones [3]. This is achieved with the aid of auxiliary fields that extend the phase space in a convenient way to transform the second-class into first class constraints.

Originally the BFFT method was formulated in a way that the first class constraints satisfy an Abelian algebra. Banerjee *et al* [4], studying the non-Abelian Proca model, have adapted the BFFT method in order that the first class constraints obey a non-Abelian algebra. This possibility pointed out by Banerjee *et al.* leads to a richer structure compared with the usual BFFT case. Recently, the Abelian and non-Abelian BFFT extension [4] were used to transform the $SU(2)$ Skyrme model in an Abelian and non-Abelian gauge theories [5], respectively.

In this work we use this non-Abelian extension of BFFT formalism to convert the second class constraints of the $O(N)$ nonlinear sigma model into first class ones. The corresponding Hamiltonian is derived solving a differential equation in an unknown function of the auxiliary fields. The Lagrangian that leads to this new theory is also derived.

The functional Schrödinger representation has recently been systematically used in order to quantize different field theories, including gravity [6–9]. Many theoretical as well as some physical predictions have been derived, for different theories, from the wave-functionals obtained so far. One example of an important theoretical feature of gauge theories established in the context of the functional Schrödinger representation, without any 'instanton' approximation, is the so-called vacuum angle [6]. On the other hand, from the wave-functional of the quantum Schwarzschild-de Sitter black hole one is able to predict how it depends on the mass and cosmological constants [9].

Here, we quantize the first class O(2) nonlinear sigma model using the functional Schrödinger representation. Since this theory is constrained we apply the so-called "reduced phase-space" quantization procedure [10]. The crucial step, in this section, is the polar transformation from the original fields (ϕ_1, ϕ_2) to new fields (R, Θ). This transformation is naturally suggested by the O(2) symmetry of the theory. In terms of (R, Θ) the functional Schrödinger equation is greatly simplified. From this equation it is clear that the energy of the theory is divided in two parts: a radial one (depends only on R) and an angular one (depends only on Θ). With an appropriated suggestion for the ground state energy, we explicitly compute the ground state wave-functional and indicate how to calculate the excited states wave-functionals.

The field-antifield formalism, created by I. Batalin and G. Vilkovisky (BV method) [11], has been used successfully to quantize the most difficult gauge theories such as supergravity theories and topological field theories in the Lagrangian framework [12–14]. The BV method comprises the Faddeev-Popov quantization [15] and has the BRST symmetry as its fundamental principle [16]. The method has introduced the definition of the antifields which are the sources of the BRST transformation, i.e., for each field we have an antifield canonically conjugated in terms of the antibracket operation. With the fields, the antifields and the BRST transformation we can construct the classical BV action. A

mathematical ingredient, called the antibracket, helps us to construct the fundamental equation of the formalism at the classical level, the so-called master equation. We may mention an extension of the BV formalism where one works in the BRST superspace. There, the main ingredient is the definition of the superfields. The details can be found in [17].

At the quantum level, we can define another mathematical operator, the Δ operator, which is a second order differential operator. From the classical BV action and its local counter terms, we can construct the quantum BV action and analogously to the classical case, the quantum master equation.

The quantization is performed via the usual functional integration through the definition of the generating functional and with the help of the well known Legendre transformation with respect to the sources JA. When it is not possible to find a solution to the quantum master equation we can say that the theory has an anomaly. The presence of a $\delta(0)$ term in the Δ operation demand a method to treat this divergence conveniently. There are various methods to regularize the theory such as Pauli-Villars [18], BPHZ [19,20] and dimensional regularization [21]. Newly, the non-local regularization (NLR) [22,23] coupled with the field-antifield formalism [24–26] has been developed. The success of the NLR is based on its power to compute the anomaly on higher-loops. Three of us, recently, have calculated the one-loop anomaly of the $SU(2)$ Skyrme model, using the NRL formalism [27].

In this work we analyze the symmetries disclosed in the conversion method that are destroyed at the quantum level. Inside the field-antifield point of view, this so-called anomaly (as we said above) is also important because it brings an impossibility to solve the quantum master equation. In the computation of the one-loop anomaly of the $O(N)$ nonlinear sigma model, we use the BV quantization coupled to NLR.

The paper is organized as follows: in section 2 we use the BFFT procedure to describe the $O(N)$ nonlinear sigma model as a gauge theory; in section 3 the quantization of the model following the Schrödinger functional method is accomplished for $N = 2$. With the gauge theory we compute the BRST transformations and calculate the one loop anomaly using the above mentioned non-local BV formalism. This is done in section 4. The final considerations are in section 5.

II. O(N) NONLINEAR SIGMA MODEL: NON-ABELIAN BFFT EMBEDDING

The $O(N)$ nonlinear sigma model is described by the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\,\partial_\mu\phi^a\partial^\mu\phi^a + \frac{1}{2}\,\lambda\,(\phi^a\phi^a - 1), \tag{2.1}$$

where the $\mu = 0, 1$ and a is an index related to the $O(N)$ symmetry group. The second class constraints of the theory are

$$T_1 = \phi^a\phi^a - 1,$$
$$T_2 = \phi^a\pi_a. \tag{2.2}$$

Following the prescription of the BFFT method with a non-Abelian algebra, the new first class constraints are given by

$$\tilde{T}_1 = \phi^a \phi^a - 1 + \eta^1, \tag{2.3}$$

$$\tilde{T}_2 = \phi^a \pi_a - \eta^2 + \eta^1 \eta^2, \tag{2.4}$$

where η^1 and η^2 are auxiliary fields that satisfy the following algebra

$$\{\eta^a, \eta^b\} = 2\delta(x - y). \tag{2.5}$$

The first class constraint algebra is

$$\{\tilde{T}_1(x),\, \tilde{T}_1(y)\} = 0,$$
$$\{\tilde{T}_1(x),\, \tilde{T}_2(y)\} = 2\,\tilde{T}_1(x)\,\delta(x - y),$$
$$\{\tilde{T}_2(x),\, \tilde{T}_2(y)\} = 0. \tag{2.6}$$

Our next step is the calculation of the extended canonical Hamiltonian density. The canonical Hamiltonian density is

$$\mathcal{H}_c = \frac{1}{2}\,\pi_a \pi_a - \frac{1}{2}\,\partial_i \phi^a \partial^i \phi^a - \frac{1}{2}\,\lambda\left(\phi^a \phi^a - 1\right). \tag{2.7}$$

In order to derive the corresponding Hamiltonian in the extended phase space, we consider [5]

$$\tilde{H}_c = \int dx \left[\frac{1}{2}\,\pi_a \pi_a (1 - \eta^1) - \phi^a \pi_a \eta^2 (1 - \eta^1) \right.$$
$$\left. + \frac{1}{2}\phi^a \phi^a \eta^2 \eta^2 (1 - \eta^1) - \frac{1}{2}\,\phi^a \partial_i \partial_i \phi^a f(\eta^1) \right], \tag{2.8}$$

where $f(\eta^1)$ is an unknown function of the auxiliary field η^1. In order to obtain $f(\eta^1)$, let us demand that

$$\{\tilde{T}_\alpha, \tilde{H}_c\} = 0, \quad \alpha = 1, 2. \tag{2.9}$$

We note that this expression is evident for $\alpha = 1$. From the equation for $\alpha = 2$, we get

$$\frac{f\prime(\eta^1)}{f(\eta^1)} = \frac{1}{1 - \eta^1}, \tag{2.10}$$

where the prime means derivative with respect to the auxiliary field η^1. From Eq.(2.10) we have

$$f(\eta^1) = \frac{1}{1 - \eta^1}. \tag{2.11}$$

Substituting expression (2.11) into (2.8), we obtain

$$\tilde{H}_c = \int dx \left[\frac{1}{2} \pi_a \pi_a (1 - \eta^1) - \phi^a \pi_a \eta^2 (1 - \eta^1) \right.$$
$$\left. + \frac{1}{2} \phi^a \phi^a \eta^2 \eta^2 (1 - \eta^1) - \frac{1}{2} \phi^a \partial_i \partial_i \phi^a \frac{1}{1 - \eta^1} \right]. \tag{2.12}$$

In order to embed the $O(N)$ nonlinear sigma model in a non-Abelian gauge theory we use the equivalent first class Hamiltonian, which differs from the involutive Hamiltonian (2.12) by the addition of a term proportional to the first class constraint $\tilde{T}_2$, as follows,

$$\tilde{H}_{c_1} = \int dx \left[\frac{1}{2} \pi_a \pi_a (1 - \eta^1) - \phi^a \pi_a \eta^2 (1 - \eta^1) \right.$$
$$\left. + \frac{1}{2} \phi^a \phi^a \eta^2 \eta^2 (1 - \eta^1) - \frac{1}{2} \phi^a \partial_i \partial_i \phi^a \frac{1}{1 - \eta^1} + \eta^2 (\phi^a \pi_a - \eta^2 + \eta^1 \eta^2) \right]. \tag{2.13}$$

We note that this Hamiltonian satisfies the first class Poisson algebra

$$\{\tilde{T}_1, \tilde{H}_{c_1}\} = 2\tilde{T}_2 + 2\eta^2 \tilde{T}_1,$$
$$\{\tilde{T}_2, \tilde{H}_{c_1}\} = 2\eta^2 \tilde{T}_2. \tag{2.14}$$

Finally, we look for the Lagrangian that leads to this new theory. A consistent way of doing this is by means of the constrained path integral formalism, where the Faddeev procedure [28] has to be used.

In the Hamiltonian formalism, let us identify the new variables η^a as a canonically conjugate pair (φ, π_φ),

$$\eta^1 \to 2\varphi\,, \tag{2.15}$$

and

$$\eta^2 \to \pi_\varphi. \tag{2.16}$$

They satisfy the relation (2.5). Then, the general expression for the vacuum functional reads

$$Z = N \int [d\mu] \exp\{i \int dx dt [\dot{\phi}^a \pi_a + \dot{\varphi} \pi_\varphi - \tilde{H}_{c_1}]\}, \tag{2.17}$$

with the measure $[d\mu]$ given by

$$[d\mu] = [d\phi^a][d\pi_a][d\varphi][d\pi_\varphi]|det\{,\}|\delta(\phi^a \phi^a - 1 + 2\varphi)\delta(\phi^a \pi_a - \pi_\varphi + 2\varphi \pi_\varphi) \prod_\alpha \delta(\tilde{\Lambda}_\alpha)\,, \tag{2.18}$$

where $\tilde{\Lambda}_\alpha$ are the gauge fixing conditions corresponding to the first class constraints $\tilde{T}_\alpha$ and the term $|det\{,\}|$ represents the determinant of all constraints of the theory, including the gauge-fixing ones. The quantity N that appears in (2.17) is the usual normalization

factor. Using the delta functions $\delta(\phi^a\phi^a-1+2\varphi)$, $\delta(\phi^a\pi_a-\pi_\varphi(1-2\varphi))$ and exponentiating the last one with the Fourier variable ξ, we obtain

$$
\begin{aligned}
Z = N \int [d\phi^a][d\pi_a][d\varphi][d\pi_\varphi][d\xi]|det\{,\}|\,\delta(\phi^i\phi^i - 1 + 2\varphi) \\
\prod_\alpha \delta(\tilde{\Lambda}_\alpha) \exp\{i \int dxdt[\dot{\phi}^a\pi_a + \dot{\varphi}\pi_\varphi - \frac{1}{2}\pi_a\pi_a\phi^b\phi^b + \pi_a\pi_a\phi^b\phi^b\pi_\varphi \\
- \frac{1}{2}(\phi^a\phi^a)^2\pi_\varphi\pi_\varphi + \xi\phi^a\pi_a - \xi\phi^a\phi^a\pi_\varphi + \frac{1}{2}\phi^a\partial_i\partial_i\phi^a\frac{1}{1-2\varphi}]\} \ .
\end{aligned} \tag{2.19}
$$

Performing the integration over the momenta and the variable ξ, we obtain

$$
\begin{aligned}
Z = N \int [d\phi^i][d\varphi]\frac{1}{(\phi^a\phi^a)^{1/2}}\,\delta(\phi^a\phi^a - 1 + 2\varphi)\,\delta(2\phi^a\dot{\phi}^a + 2\dot{\varphi}) \\
\prod_\alpha \delta(\tilde{\Lambda}_\alpha)|det\{,\}|\exp\{i \int dxdt[\frac{\dot{\phi}^i\dot{\phi}^i}{\phi^i\phi^i} - \frac{1}{2}\frac{\dot{\varphi}\dot{\varphi}}{(\phi^i\phi^i)^2} + \frac{1}{2}\phi^a\partial_i\partial_i\phi^a\frac{1}{1-2\varphi}]\}.
\end{aligned} \tag{2.20}
$$

The new delta function that appears into the expression (2.20) was obtained after integration over ξ. We notice that it does not represent any new restriction over the coordinates of the theory and leads to a consistency condition on the constraint $\tilde{T}_1$. From the vacuum functional (2.20), we identify the extended Lagrangian density

$$
\tilde{\mathcal{L}} = \frac{1}{2}\frac{\dot{\phi}^a\dot{\phi}^a}{1-2\varphi} + \frac{1}{2}\phi^a\partial_i\partial_i\phi^a\frac{1}{1-2\varphi} - \frac{1}{2}\frac{\dot{\varphi}\dot{\varphi}}{(1-2\varphi)^2} \ . \tag{2.21}
$$

In this embedding, with the choice of the non-Abelian algebras (2.6) and (2.14), we notice that in the expression of the extended Lagrangian (2.21) there is not a Liouville term in the auxiliary fields as in the reference [29]. The reason for this difference was the choice of another non-Abelian algebra in [29] .

III. REDUCED PHASE-SPACE QUANTIZATION IN A FUNCTIONAL SCHRÖDINGER REPRESENTATION.

In the present section we quantize the $O(N)$ nonlinear sigma model, written as a non-Abelian gauge theory. We canonically quantize the theory in a functional Schrödinger representation [6–8]. Therefore, as the fundamental equations representing our theory we take the constraints (2.3) and (2.4) and the Hamiltonian (2.13), all of them written in terms of the conjugated pair (φ, π_φ) (2.15) and (2.16).

Due to the presence of the constraints, we have to choose among the different procedures to canonically quantize a constrained theory. Here, we use the so-called 'reduced phase-space' quantization [10]. It means that, we have to impose classically the constraints, which reduces the theory to its physical degrees of freedom. Then, we re-write

the Hamiltonian $(\tilde{H}_c)$ in terms of these physical degrees of freedom. Finally, we canonically quantize this reduced Hamiltonian (H_c^r) in a functional Schrödinger representation.

We start imposing, classically, the constraints (2.3) and (2.4), which gives,

$$\phi^a \phi^a - 1 + 2\varphi = 0 \,, \tag{3.1}$$

$$\phi^a \pi_a - \pi_\varphi + 2\varphi \pi_\varphi = 0 \,. \tag{3.2}$$

Now, using the above equations (3.1) and (3.2), we express the field φ and its conjugated momentum π_φ in terms of the fields ϕ^a and their conjugated momenta π_a. Next, we introduce the expression relating (φ, π_φ) with (ϕ^a, π_a) in the $\tilde{H}_c$ (2.13), obtaining in this way the following reduced Hamiltonian,

$$H_c^r = \int \left[(\pi_a \pi_a)(\phi^b \phi^b) - (\phi^a \pi_a)(\phi^b \pi_b) - \phi^a \partial_i \partial_i \phi^a \frac{1}{\phi^b \phi^b} \right] dx \,. \tag{3.3}$$

It is important to notice that in the derivation of the above expression of H_c^r (3.3), from $\tilde{H}_c$ (2.13), we have also set to zero the last term of H_c. As one can see this term is proportional to the constraint (3.2).

In order to simplify our treatment, we restrict our attention to the case of two fields $(N = 2)$. Therefore, we re-write H_c^r (3.3) in terms of the fields ϕ_1 and ϕ_2 and their respective momenta π_1 and π_2, as,

$$H_c^r = \frac{1}{2} \int \left[(\pi_1 \phi_2 - \pi_2 \phi_1)^2 - \frac{1}{\phi_1^2 + \phi_2^2} (\phi_1 \partial_i \partial_i \phi_1 + \phi_2 \partial_i \partial_i \phi_2) \right] dx \,. \tag{3.4}$$

Our next step is the quantization, in the functional Schrödinger representation, of the reduced theory described by H_c^r (3.4).

We start considering ϕ_1 , ϕ_2 , π_1 and π_2 as quantum operators, it means that in the fields basis the momenta is replaced by the following functional derivatives,

$$\pi_1(x) \rightarrow -i \frac{\delta}{\delta \phi_1(x)} \quad , \quad \pi_2(x) \rightarrow -i \frac{\delta}{\delta \phi_2} \,, \tag{3.5}$$

where we have set $\hbar$ equal to one.

In general, the states of the theory are given by time-dependent functionals of the fields, namely,

$$\Psi = \Psi[\phi_1, \phi_2, t] \,. \tag{3.6}$$

This wave-functional Ψ (3.6) satisfies the Schrödinger equation,

$$i \frac{\partial}{\partial t} \Psi[\phi_1, \phi_2, t] = \hat{H}_c^r [\phi_1, \phi_2, t] \,, \tag{3.7}$$

which is a functional differential equation because $\hat{H}_c^r$, with the aid of (3.4) and (3.5), is given by,

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$$\hat{H}_c^r = \frac{1}{2} \int \left[\left(-i\frac{\delta}{\delta\phi_1}\phi_2 + i\frac{\delta}{\delta\phi_2}\phi_1 \right)^2 - \frac{1}{\phi_1^2 + \phi_2^2} \left(\phi_1 \partial_i \partial_i \phi_1 + \phi_2 \partial_i \partial_i \phi_2 \right) \right] dx . \qquad (3.8)$$

Observing the $O(2)$ symmetry of our sigma nonlinear model, it seems interesting, to re-write $\hat{H}_c^r$ in terms of a new pair of fields (R, Θ), related to the old ones (ϕ_1, ϕ_2), by,

$$\phi_1 = R\sin\Theta \quad \text{and} \quad \phi_2 = R\cos\Theta . \qquad (3.9)$$

In terms of the new fields (R, Θ) and their respective functional derivatives, which may be derived from (3.9), $\hat{H}_c^r$ is written as,

$$\hat{H}_c^r = \frac{1}{2} \int \left[-\frac{\delta^2}{\delta\Theta^2} - \Theta\partial_i\partial_i\Theta - \frac{1}{R}\partial_i\partial_i R \right] dx . \qquad (3.10)$$

It is important to mention that we have solved the factor-ordering ambiguities in $\hat{H}_c^r$ (3.10), by using the so-called 'symmetric factor-ordering' [30].

Since $\hat{H}_c^r$ does not explicitly depend on time, we may separate out the time dependence of the wave-functional, now given in terms of R and Θ ($\Psi[R, \Theta, t]$), and write,

$$\Psi[R, \Theta, t] = e^{-iEt}\Psi[R, \Theta] . \qquad (3.11)$$

$\Psi[R, \Theta]$ satisfies the time-independent Schrödinger functional equation,

$$\int \left[-\frac{\delta^2\Psi}{\delta\Theta^2} - \Theta\partial_i\partial_i\Theta\Psi - \frac{1}{R}\partial_i\partial_i R\Psi \right] dx = 2E\Psi . \qquad (3.12)$$

Following [7], in order to find the ground state or vacuum wave-functional, $\Psi_0[R, \Theta]$, we write the following ansatz for $\Psi_0[R, \Theta]$,

$$\Psi_0[R, \Theta] = \eta\exp\{-G[R, \Theta]\} . \qquad (3.13)$$

Introducing the ansatz (3.13) in equation (3.12), we obtain the below equation for $G[R, \Theta]$,

$$\int \left[\frac{\delta^2 G}{\delta\Theta^2} - \left(\frac{\delta G}{\delta\Theta} \right)^2 - \Theta\partial_i\partial_i\Theta - \frac{1}{R}\partial_i\partial_i R \right] dx = 2E . \qquad (3.14)$$

This equation naturally separates in two parts, one that depends solely on R and another that depends on R and Θ, through $G[R, \Theta]$. Such that we may re-write (3.14) as,

$$\int \alpha(x)dx + \int \beta(x)dx = 2E , \qquad (3.15)$$

where,

$$-\frac{1}{R}\partial_i\partial_i R = \beta(x) , \qquad (3.16)$$

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and

$$\int \left[\frac{\delta^2 G}{\delta \Theta^2} - \left(\frac{\delta G}{\delta \Theta} \right)^2 - \Theta \partial_i \partial_i \Theta \right] dx \; = \; \int \alpha(x) dx \,. \tag{3.17}$$

Equation (3.15) may be interpreted as saying that the energy of the system is divided in two parts. The first part $\int \beta(x) dx$ is entirely determined by R from (3.16), we call it E_R. The second one $\int \alpha(x) dx$ is determined by a functional that might depend on R and Θ, we call it E_Θ.

For a given function $\beta(x)$ one finds, with the aid of (3.16) and appropriated boundary conditions upon $R(x)$, one function $R(x)$. Therefore, we may see that $R(x)$ will not be allowed to be a generic function in the function space.

It is important to notice that $\beta(x)$ has to satisfy certain conditions such as finiteness and positiveness of $\int \beta(x) dx$. For a positive $\int \beta(x) dx$, we may define the ground sate of E_R to be the one where $\beta(x) = 0$. States with $\beta(x) \neq 0$ would represent the excited states.

Equation (3.17) is well-known in the literature of quantization in a functional Schrödinger representation. It is the equation for a massless scalar field [7]. The functional $G[R, \Theta]$ which satisfies (3.16), for the present situation, has the following expression,

$$G[R, \Theta] \; = \; \int dy dx \, \Theta(y) g(y, z) \Theta(z) + \int dz \left(-\frac{1}{R(z)} \partial_i \partial_i R(z) \right) , \tag{3.18}$$

where the last term on the right hand side is simply $\int \beta(z) dz$.

Introducing (3.18) in (3.17), we may obtain the explicit expression for $g(y, z)$,

$$g(y, z) \; = \; \frac{1}{2} \int \frac{dk}{2\pi} k e^{ik(y-z)} \,, \tag{3.19}$$

and the ground state energy of E_Θ. This energy is derived by computing $\int \alpha(x) dx$, which gives,

$$\int \alpha(x) dx \; = \; \int g(x, x) dx \; = \; \frac{1}{2} \int \frac{dk}{2\pi} k \int dx \; = \; \frac{1}{2} \int dk k \delta(0) \,. \tag{3.20}$$

It agrees with the result obtained in the operator representation [7].

An important result comes from (3.20). $G[R, \Theta]$(3.18), has two components: one that depends on Θ (G_Θ) and another that depends on R (G_R). From (3.20), it is clear that E_Θ, $\int \alpha(x) dx$, is entirely determined by G_Θ. Therefore it does not depend upon R.

Now, we set $\beta(x) = 0$ in (3.18), accordingly to our suggestion to the ground state energy of E_R, and combine the resulting expression with (3.13). Then, we may obtain the normalized ground state wave-functional of the Fourier transform of $\Theta(x)$ ($\Psi_0[\tilde{\Theta}(k)]$) as [7],

$$\Psi_0[\tilde{\Theta}(k)] \; = \; \prod_k \left(\frac{k}{\pi} \right)^{1/4} \exp \left(\frac{-1}{4\pi} k \tilde{\Theta}^2(k) \right) \,. \tag{3.21}$$

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This is just the infinite product of harmonic oscillators ground state wave-functions, one wave-function for each k.

For the excited states we have $\beta(x) \neq 0$ which, from (3.18), would introduce a R dependence in the wave-functionals. As a typical example we may write,

$$\Psi_1[R, \Theta] = \frac{k_1^{1/2}}{\pi} \int dy\, e^{-ik_1 y} \Theta(y) \Psi_0[\Theta] \exp\left[\int dz \frac{1}{R(z)} \partial_i \partial_i R(z)\right] . \tag{3.22}$$

It represents the wave-functional associated with the first excited state of E_Θ [7], and a generic excited state of E_R.

As a matter of completeness we would like to mention the reference [31], where the functional Schrödinger representation was first applied to the study of $O(N)$ nonlinear sigma models. We may identify two main differences between [31] and the present work. Firstly, we have re-written the theory as a non-Abelian gauge theory, and in [31] it was treated as a second-class system. Secondly, we have explicitly solved the functional Schrödinger equation and found the ground state as well as the excited wave-functionals for $N = 2$. In [31] it was computed the expected value of the Hamiltonian using a trial wave-functional, in the large N limit.

IV. THE ONE-LOOP ANOMALY

In this section we follow the standard references about the BV formalism [11–14] and the NLR [22,23] coupled to the field-antifield procedure [17,24–26]. All the details of the theory involved in the following calculation of the $O(N)$ nonlinear sigma model anomaly can be found in those papers.

The first class constraint (2.3), written in terms of φ of equation (2.15), tell us that

$$\varphi = 1 - \phi^a \phi^a \ , \tag{4.1}$$

so that,

$$\dot{\varphi} = -\phi^a \dot{\phi}^a \ . \tag{4.2}$$

Substituting this in (2.21) we have now that

$$\mathcal{S} = \frac{1}{2} \int dx\, dt \left[\frac{\dot{\phi}^a \dot{\phi}^a + \phi^a \partial_i \partial_i \phi^a}{\phi^a \phi^a} - \frac{(\dot{\phi}^a \dot{\phi}^a)^2}{(\phi^a \phi^a)^2}\right] \ , \tag{4.3}$$

This action, as we can easily see, has a problem of non-locality, which can be solved expanding the terms,

$$\mathcal{S} = \frac{1}{2} \int dt \left\{\frac{\dot{\phi}^a \dot{\phi}^a + \phi^a \partial_i \partial_i \phi^a}{[1 - (1 - \phi^a \phi^a)]} - \frac{(\dot{\phi}^a \dot{\phi}^a)^2}{[1 - (1 - \phi^a \phi^a)]^2}\right\}$$

$$= \frac{1}{2} \left(\dot{\phi}^a \dot{\phi}^a + \phi^a \partial_i \partial_i \phi^a \right) \sum_{n=0}^{\infty} (1 - \phi^a \phi^a)^n$$

$$- \frac{1}{2} (\dot{\phi}^a \dot{\phi}^a)^2 \sum_{n=0}^{\infty} (n+1)(1 - \phi^a \phi^a)^n \quad . \tag{4.4}$$

After a simple calculation, we can say that this action is invariant under the BRST transformations given by

$$\delta \phi^a = c \, \phi^a \, , \tag{4.5}$$

and

$$\delta c = 0 \quad , \tag{4.6}$$

where $c = c^a \, T^a$ and $tr \, (T^a \, T^a) = \frac{1}{2}$.

Now we can construct the BV action,

$$S_{BV} = \frac{1}{2} \int dt \, dx \left\{ \left(\dot{\phi}^a \dot{\phi}^a + \phi^a \partial_i \partial_i \phi^a \right) \sum_{n=0}^{\infty} (1 - \phi^a \phi^a)^n \right.$$

$$\left. - (\dot{\phi}^a \dot{\phi}^a)^2 \sum_{n=0}^{\infty} (n+1)(1 - \phi^a \phi^a)^n + \phi^{a*} c \, \phi^a \right\} . \tag{4.7}$$

The kinetic part of the action (4.3) (i.e., with $n = 0$) after an integration by parts is,

$$F = \frac{1}{2} \int d \, t \, [\, \dot{\phi}^a \, \dot{\phi}^a \,]$$

$$= \frac{1}{2} \int dt [\, -\phi^a \, \partial_0^2 \, \phi^a \,] \quad . \tag{4.8}$$

Hence, the kinetic term has the form

$$F = \frac{1}{2} \phi^a (-\partial_0^2) \, \phi^a$$

$$\implies \mathcal{F}_{AB} = -\partial_0^2 \, \delta_{AB}. \tag{4.9}$$

The regulator, a second order differential operator, can be chosen as

$$\mathcal{R}_B^A = \partial_0^2 \, \delta^A{}_B$$

$$\implies T = -1 \quad . \tag{4.10}$$

where T, as required, is clearly an arbitrary non-singular matrix.

The smearing operator has the form,

$$\epsilon^A{}_B = exp \left(\frac{\partial_0^2}{2 \, \Lambda^2} \right) \, \delta^A{}_B \quad .$$

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In the NLR scheme the shadow kinetic operator is

$$\mathcal{O}_{AB}^{-1} = \left(\frac{\mathcal{F}}{\epsilon^2 - 1}\right)_{AB} = \left(\frac{-\partial_0^2}{\epsilon^2 - 1}\right)_{AB}, \tag{4.11}$$

where

$$\mathcal{O}^{AB} = -\frac{\epsilon^2 - 1}{\partial_0^2} = -\int_0^1 \frac{d\tau}{\Lambda^2} \exp\left(\tau \frac{\partial_0^2}{\Lambda^2}\right). \tag{4.12}$$

Using the definitions of $S^A{}_B$ and $\mathcal{I}_{AB}$, we can show that

$$S^\phi{}_\phi = c, \tag{4.13}$$

$$\mathcal{I}_{\phi\phi} = -\partial_0^2 + \frac{\partial_0^2 + \partial_i^2}{\phi^a \phi^a} - \frac{4\phi^a \dot{\phi}^a \partial_0 + 3\phi^a \partial_i^2 \phi^a + 2\phi^a \phi^a \partial_i^2 + \dot{\phi}^a \dot{\phi}^a + \partial_0}{(\phi^a \phi^a)^2}$$

$$+ \frac{2\phi^a \phi^a (\dot{\phi}^a \dot{\phi}^a + \phi^a \partial_i \partial_i \phi^a) + 3\phi^a \dot{\phi}^a + 2\phi^a \phi^a \partial_0}{(\phi^a \phi^a)^3}$$

$$- 2\frac{\phi^a \phi^a (\phi^a \dot{\phi}^a)}{(\phi^a \phi^a)^4}. \tag{4.14}$$

Finally, the anomaly can be computed as we know

$$\mathcal{A} = (\Delta S)_R$$
$$= \lim_{\Lambda^2 \to \infty} \{ Tr[\epsilon^2 S^A{}_B] + Tr[\epsilon^2 S^A{}_D \mathcal{O}^{DC} \mathcal{I}_{CB}] \}. \tag{4.15}$$

Computing each term, we have that the only non-zero integral is the one coming from the second term in $\mathcal{I}_{\phi\phi}$ (4.13). Note that, differently from [27], now we are in a two dimensional space. Let us write only the main steps of this calculation. So,

$$\lim_{\Lambda^2 \to \infty} \left[\epsilon^2 c \int dt\, dx\, \mathcal{O} \frac{(\partial_0^2 + \partial_i^2)}{\phi^a \phi^a} \right]$$

$$= \lim_{\Lambda^2 \to \infty} \left[\epsilon^2 c \int dt\, dx \int \frac{d^2 k}{(2\pi)^2} e^{-ikx} \mathcal{O} \frac{(\partial_0^2 + \partial_i^2)}{\phi^a \phi^a} \exp\left(\frac{\partial^2}{\Lambda^2}\right) e^{ikx} \right]$$

$$= \lim_{\Lambda^2 \to \infty} \left[\epsilon^2 c \int dt\, dx \int \frac{d^2 k}{(2\pi)^2} e^{-ikx} \int_0^1 \left(-\frac{d\tau}{\Lambda^2}\right) \exp\left(\tau \frac{\partial_0^2}{\Lambda^2}\right) \frac{(\partial_0^2 + \partial_i^2)}{\phi^a \phi^a} \exp\left(\frac{\partial^2}{\Lambda^2}\right) e^{ikx} \right]$$

$$= \lim_{\Lambda^2 \to \infty} \left[\epsilon^2 c \int dt\, dx \int_0^1 \left(-\frac{d\tau}{\Lambda^2}\right) \exp\left(\tau \frac{\partial_0^2}{\Lambda^2}\right) \frac{1}{\phi^a \phi^a} \right.$$

$$\left. \times \int \frac{dk_0\, dk_1}{(2\pi)^2} (-k_0^2 - k_1^2) \exp\left(\frac{-k_0^2 + k_1^2}{\Lambda^2}\right) \right]$$

$$= \lim_{\Lambda^2 \to \infty} \left[\frac{\pi c}{4} \epsilon^2 \int dt\, dx \int_0^1 d\tau \left(1 + \tau \frac{\partial_0^2}{\Lambda^4}\right) \frac{1}{\phi^a \phi^a} \right]$$

$$= \frac{\pi}{4} \int dt\, dx \frac{c}{\phi^a \phi^a}, \tag{4.16}$$

where we make two convenient reparametrizations,

$$(\tau, k) \rightarrow (\frac{\tau}{\lambda^2}, \lambda k), \tag{4.17}$$

to solve the integrals [32].

Repeating the same procedure (integration) for all the other terms of the $\mathcal{I}_{\phi\phi}$ (4.13) one can conclude that they are identically zero, as we have said above.

As we know, terms that depend only on ghosts do not have any physical meaning in the final result of the anomaly. Computing only the physical terms, the one-loop anomaly for the $O(N)$ nonlinear sigma model is the Wess-Zumino consistent expression [33],

$$\mathcal{A} = \frac{\pi}{4} \int dt \, \frac{c}{\phi^a \, \phi^a} \quad . \tag{4.18}$$

It is a new result, showing that, at the level of the BV formalism, the QME has no solution.

V. CONCLUSIONS

In this work we studied the $O(N)$ nonlinear sigma model embedding this system in a non-Abelian gauge theory. It was accomplished through an extension of the BFFT conversion method.

Following the prescription of this method, we obtained the new first-class constraints, the extended Hamiltonian and the effective Lagrangian that leads to the new theory.

Then, we quantized the first class O(2) nonlinear sigma model using the functional Schrödinger representation. Since this theory is constrained, we applied the so-called "reduced phase-space" quantization procedure. The crucial step, in this program, was the polar transformation from the original fields (ϕ_1, ϕ_2) to new fields (R, Θ). This transformation is naturally suggested by the O(2) symmetry of the theory. In terms of (R, Θ), the functional Schrödinger equation was greatly simplified. From this equation it was clear that the energy of the theory is divided in two parts: a radial one (depends only on R) and an angular one (depends only on Θ). With an appropriated suggestion for the ground state energy, we explicitly computed the ground state wave-functional and indicated how to calculate the excited states wave-functionals.

Finally, we have computed the anomaly at one-loop order of the $O(N)$ nonlinear sigma model through the introduction of its BRST transformations and consequently of all the ingredients of the field-antifield procedure.

The importance of a conversion constraint method is fundamentally to have a gauge theory. These gauge symmetries, at the classical level, give rise to conserved Noether currents. In many cases, including the one studied here, the expected value of the Noether currents are not conserved.

Based on the results obtained in [27] and in this work, we believe that the BFFT procedure of extension of the phase space affects the Wess-Zumino sector of the O(N) non-linear sigma model. At the quantum level, it causes the non conservation of the Noether currents generated by the classical gauge symmetries mentioned above.

VI. ACKNOWLEDGMENTS

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